Adaptive vs Rational Expectations

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1. Introduction

Why Models with RE

- **Economics is different**: contrary to physics, biology, and other subjects, in economics most decisions are based on the agents's expectations
- **Two competing views** in economics to deal with the formulation of expectations:
  - Backward-looking (or adaptive) expectations
  - Forward looking (or rational) expectations
- How relevant are they?
- Let us look at some simple examples.

Example 1: The Evolution of Public Debt

- Recall that the evolution of public debt as a percentage of GDP (d_t) is given by:

$$d_t = p + \left(\frac{1+r}{1+g}\right) d_{t-1}$$

- We may remember that the stability of d_t depends crucially on whether:

$$1+r > 1+g \quad , \quad 1+r < 1+g \quad , \quad 1+r = 1+g$$

- The variable d_t is called a ***pre-determined*** variable because it is determined by past values which we can observe.

Example 2: A Financial Investment

- Consider a financial asset:
  - Bought today at price P_t, and pays a dividend of D_t per period.
  - Assume a close substitute asset (e.g., a bank deposit with interest) that yields a safe rate of return given by r.
- A risk neutral investor holds the asset if both assets get the same expected rate of return

$$\frac{D_t + \mathbb{E}_t P_{t+1}}{P_t} = 1 + r$$

- Solve for P_t, and to simplify define $\phi = 1/(1+r)$, to get

$$P_t = \phi D_t + \phi \mathbb{E}_t P_{t+1}$$

- How do we solve such an equation? P_t is called a ***forward-looking variable***.

Example 3: The Cagan Model

- In 1956, Phillip Cagan published a very famous paper with the title "The Monetary Dynamics of Hyperinflation".
- The model involves the money demand (m^d):

$$p_t = \frac{a}{1+\beta} r_t + \frac{a}{1+\beta} \mathbb{E}_t p_{t+1} + \frac{1}{1+\beta} m_t^d$$

p_t, r_t are the price level and the real interest rate. The supply of money (m^s) is:

$$m_t^s = \phi + \theta m_{t-1}^s + \epsilon_t \quad , \quad |\theta| < 1$$

- The central banks sets the supply of money.
- How can we solve such model, having $E_t p_{t+1}$ in one equation?

2. Mathematics required

Solution of a Geometric Series

- Suppose we have a process that is written as:

$$s = \rho^0\phi + \rho^1\phi + \rho^2\phi + \rho^3\phi \ + \ \ldots \ = \sum_{i=0}^{\infty} \phi\rho^i$$

- It has two crucial elements:
  - First term of the series (when $i = 0$): ϕ
  - The common ratio: ρ
- The solution is given by the expression:

$$s = \frac{\text{first term}}{1 - \ \text{common ratio}} = \frac{\phi}{1-\rho} \ , \qquad |\rho| < 1$$

3. Adaptive Expectations (AE)

A Simple Rule of Thumb

- The world is extremely complex, do not try to be too clever.
- Use a simple rule of thumb to forecast the future.
- Extrapolate from what we have observed in the past.
- Try to correct the mistakes we made in the past, by doing for example:

$$P_t^e = P_{t-1}^e + \alpha \left(P_{t-1} - P_{t-1}^e \right), \quad 0 \leq \alpha \leq 1 \tag{1}$$

- This is the classical example of adaptive expectations: P stands for the price level and P_t^e stands for the expected price level.
- The parameter α gives the velocity with which agents correct past mistakes:
  - $\alpha \to 0$; past mistakes slowly corrected
  - $\alpha \to 1$; past mistakes quickly corrected

A Solution to Adaptive Expectations

- Start with the eq. (1) above

$$P_t^e = P_{t-1}^e + \alpha\left(P_{t-1} - P_{t-1}^e\right)$$

- Isolate P_{t-1} and get:

$$P_t^e = \alpha P_{t-1} + (1-\alpha)P_{t-1}^e$$

- Get rid of P_{t-1}^e by using:

$$P_{t-1}^e = \alpha P_{t-2} + (1-\alpha)P_{t-2}^e$$

- After this second step in the iteration process, we get

$$P_t^e = \alpha P_{t-1} + \alpha(1-\alpha)P_{t-2} + (1-\alpha)^2 P_{t-2}^e$$

- Next, get rid of P_{t-2}^e and obtain

$$P_t^e = \alpha P_{t-1} + \alpha(1-\alpha)P_{t-2} + \alpha(1-\alpha)^2 P_{t-3} + (1-\alpha)^3 P_{t-3}^e$$

- We have iterated ***three times*** backwards $(n = 3)$. We can stop here because there is a pattern that can be easily spotted (see **Appendix A** for details):

$$P_t^e = (1-\alpha)^3 P_{t-3}^e + \sum_{i=0}^{3-1} \alpha(1-\alpha)^i P_{t-1-i} \tag{2}$$

- If we iterate backwards n-times, the solution will be given by:

$$P_t^e = (1-\alpha)^n P_{t-n}^e + \sum_{i=0}^{n-1} \alpha(1-\alpha)^i P_{t-1-i}$$

- The stability of this process depends upon whether

$$|1-\alpha| < 1\,, \quad |1-\alpha| > 1$$

- To secure a stable solution, we have to impose

$$|1 - \alpha| < 1$$

- If we assume $0 < \alpha < 1$, the blue term above will converge to zero

$$\lim_{n \to \infty} (1 - \alpha)^n P^e_{t-n} = 0$$

- and the stable solution will be:

$$P^e_t = \sum_{i=0}^{n-1} \alpha (1 - \alpha)^i P_{t-1-i} \tag{3}$$

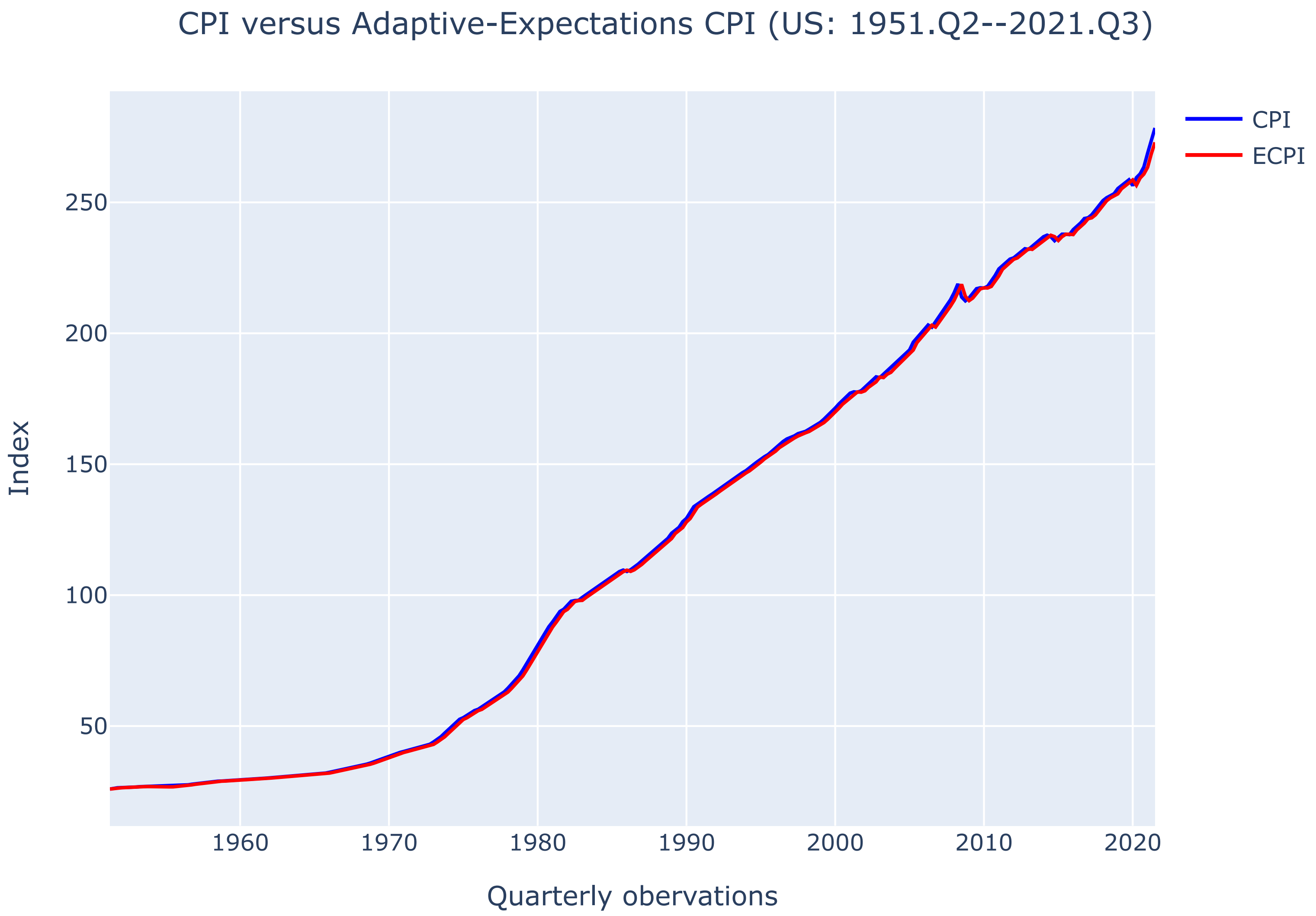
- Main message: ***expected price level depends on past price levels*** through an exponential smoothing process.

Limitations of Adaptive Expectations (AE)

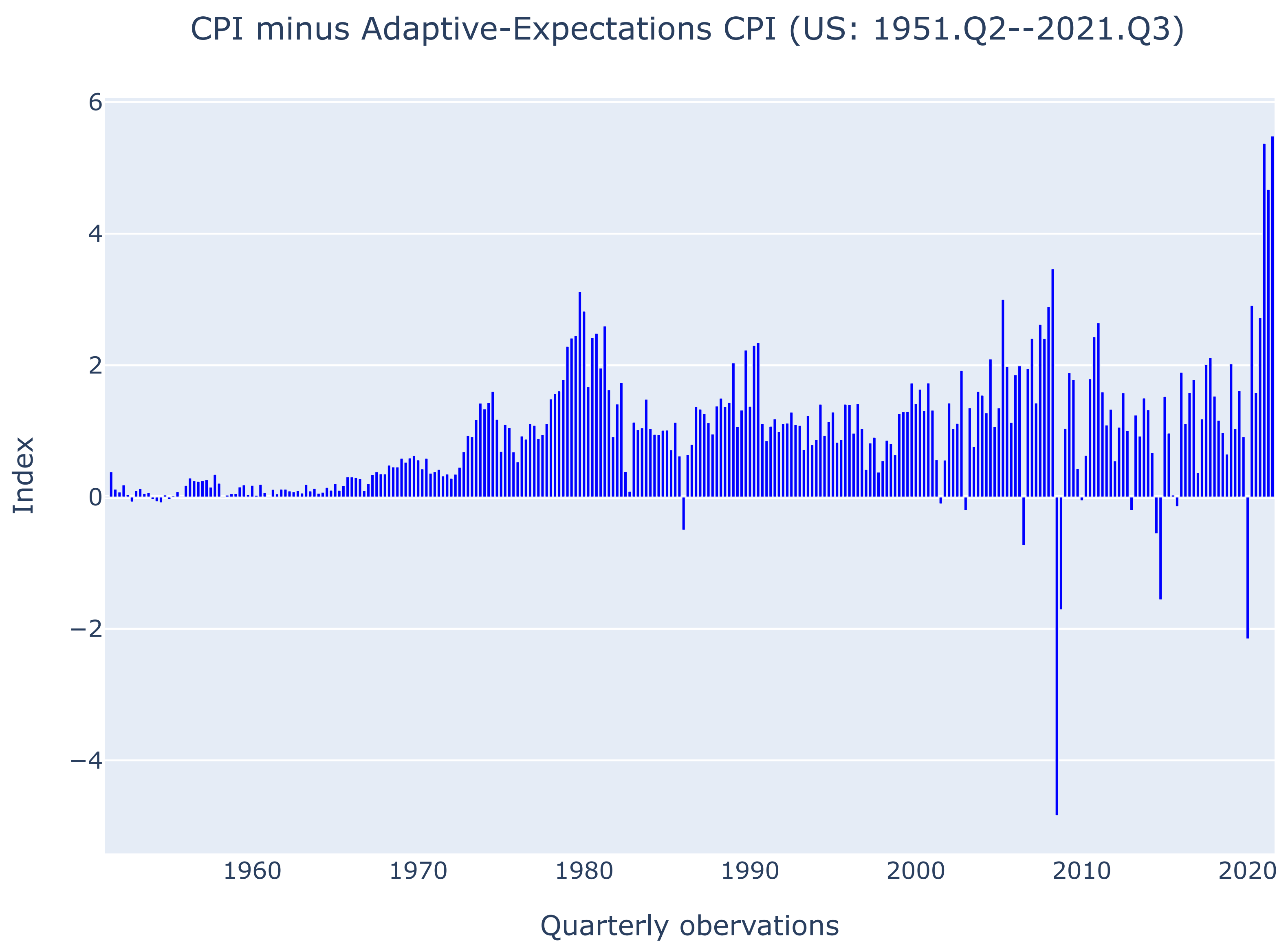
- The rule of thumb associated with AE seems to have some positive points:
  - It looks a common sense rule (don't repeat the same mistakes of the past)
  - It is easy to apply: collect data on some previous observations and make some simple calculations
- However, AE suffer from some serious drawbacks:
  - It is not logical: why should we use information only about the past? Why not information about what we expect that might happen in the future?
  - It produces biased expectations: it leads to systematic mistakes
- See next figures.

Example 1: The Price Level and AE

Consider a lag of 5 periods (quarters) and fast correction $\alpha = 0.95$. Looks great?

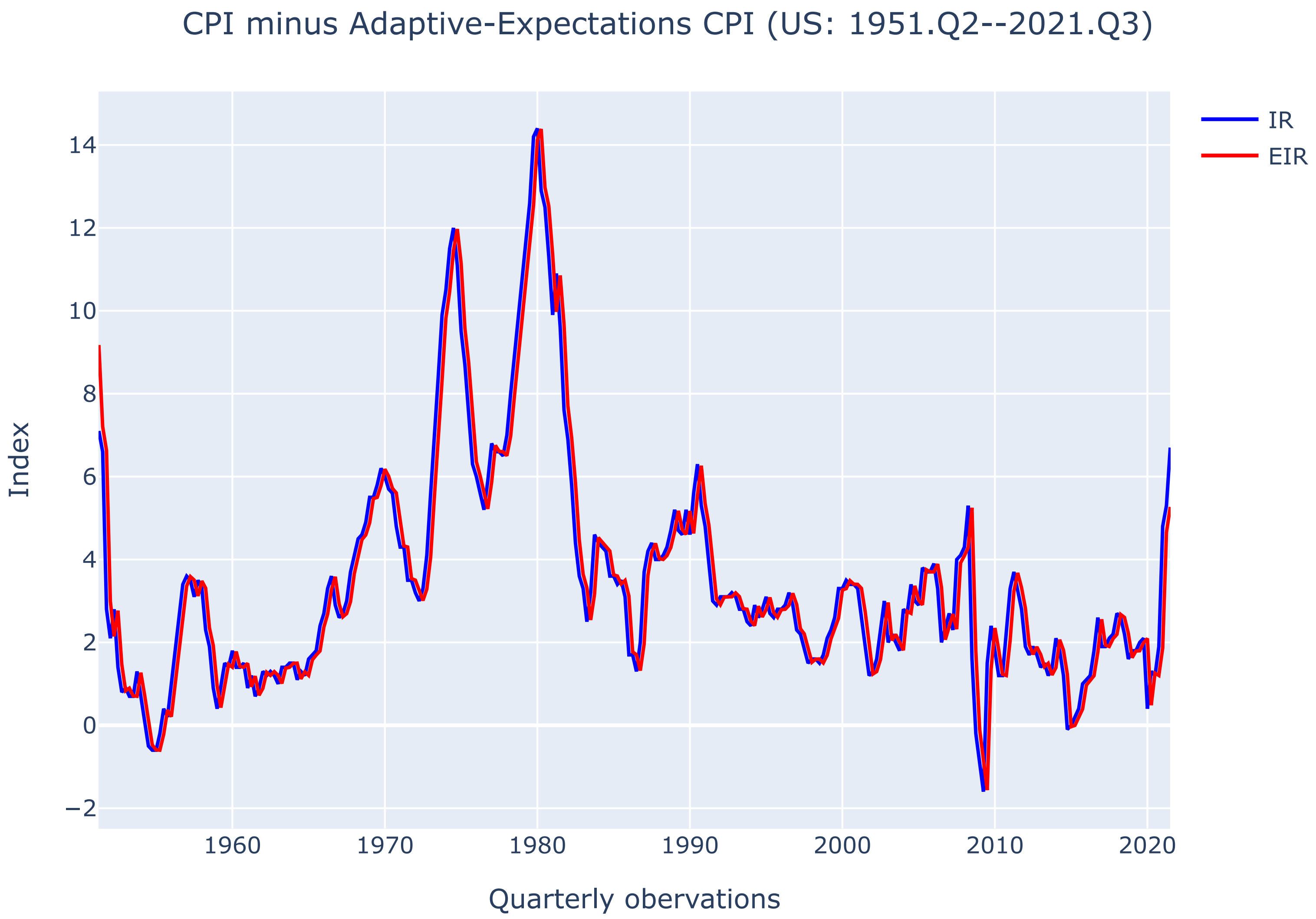


- No, it looks quite poor. The error in the forecasting exercise is large and systematic.

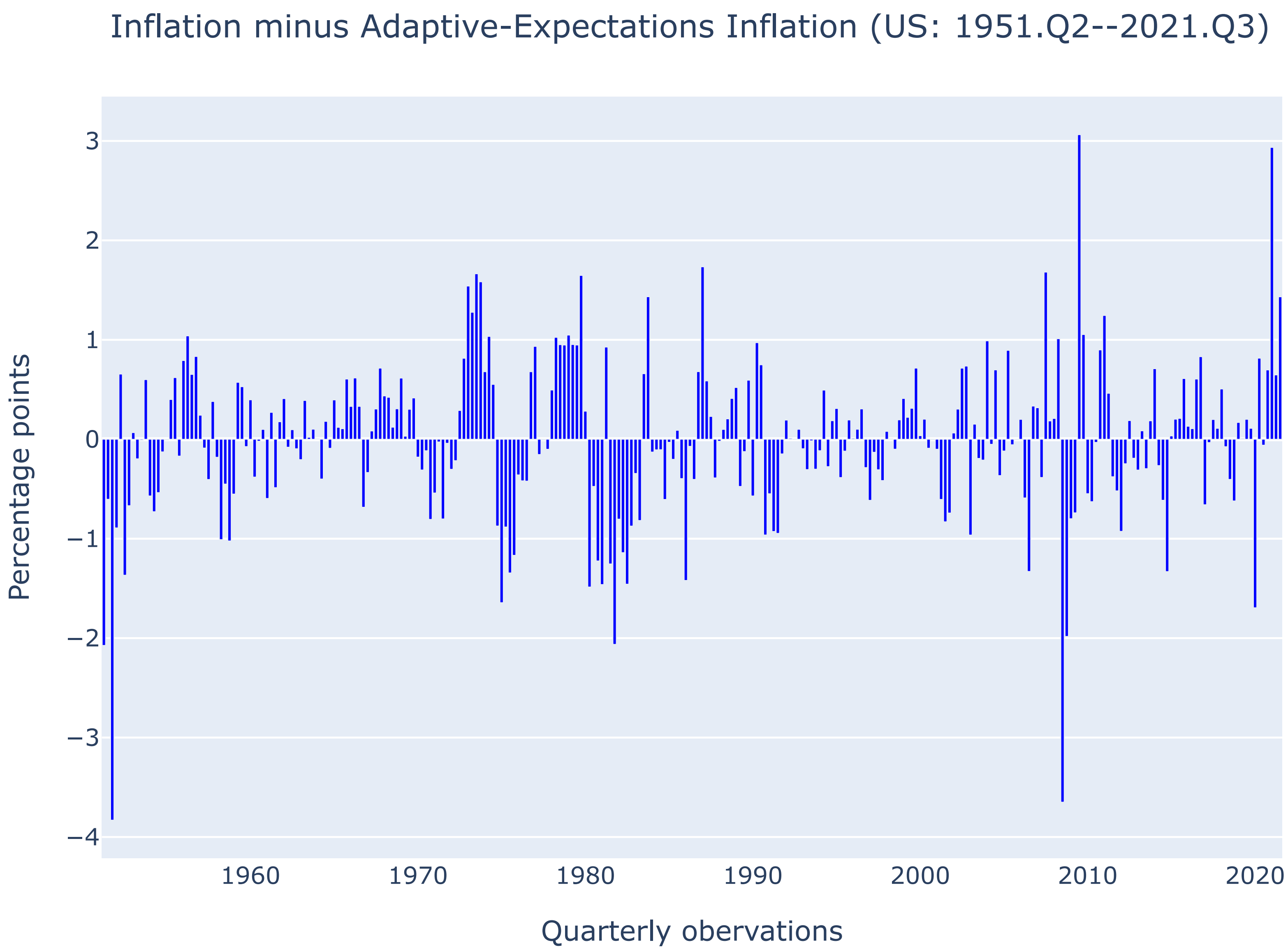


Example 2: The Rate of Inflation and AE

- Let's see what happens in the case of a stationary variable (Inflation).



- The vindication of Adaptive Expectations! The mean of the mistakes is: -0.0094989!!



Jessica James on Commodity Trading Advisors (CTAs)

CitiFX® Risk Advisory Group

Trend is popular

- 85% of CTA returns are explained by simple trend following
- The figure rises to almost 100% when carry and option trading are included
- They are without doubt the most popular systematic rule-based strategies used by overlay managers and currency alpha funds
- They may be backtested relatively easily

Jessica James was Vice President in CitiFX® Risk Advisory Group Investor Strategy, Citigroup in 2003 (when the remarks were made), and is now the Senior Quantitative Researcher in the Rates Research team at Commerzbank.

4. Rational Expectations

What are Rational Expectations?

- In modern macroeconomics, the term rational expectations means three things:
  - Agents efficiently use all publicly available information (past, present, future).
  - Agents understand the structure of the model/economy and base their expectations of variables on this knowledge.
  - Therefore, agents can forecast everything with no systematic mistakes.
- The only thing they cannot forecast are the exogenous shocks that hit the economy. These are unpredictable.
- Strong assumption: the economy's structure is complex, and nobody truly knows how everything works.

RE are the mathematical expectation of the model

- Lots of models in economics take the form:

$$y_t = x_t + a \cdot \mathbb{E}_t y_{t+1} \tag{4}$$

- It says that today's y is determined by today's x and expected value of tomorrow's y.
- What determines this expected value?
- Under the RE hypothesis, the agents understand what is in the equation and formulate expectations in a way that is consistent with it:

$$\mathbb{E}_t y_{t+1} = \mathbb{E}_t x_{t+1} + a \cdot \mathbb{E}_t \left[\mathbb{E}_{t+1} y_{t+2}\right] \tag{5}$$

- How do we solve this last equation?

The Law of Iterated Expectations

- This law states that it is not rational to expect to have a different expectation at $t+1$ for y_{t+2} than the one I have at t.
- Therefore, we get:

$$\mathbb{E}_t\left[\mathbb{E}_{t+1} y_{t+2}\right] = \mathbb{E}_t y_{t+2}$$

- Now inserting this result into eq. (5), and this one back into eq. (4), we get the 2nd iteration:

$$y_t = x_t + a \cdot \mathbb{E}_t x_{t+1} + a^2 \mathbb{E}_t y_{t+2}$$

- Getting rid of $\mathbb{E}_t y_{t+2}$, we reach the 3rd iteration:

$$y_t = x_t + a \cdot \mathbb{E}_t x_{t+1} + a^2 \mathbb{E}_t x_{t+2} + a^3 \mathbb{E}_t y_{t+3}$$

Third Iteration: Generalize

- Once arrived at the 3rd iteration we can see a pattern and generalize:

$$y_t = x_t + a \cdot \mathbb{E}_t x_{t+1} + a^2 \mathbb{E}_t x_{t+2} + \color{blue}{a^3 \mathbb{E}_t y_{t+3}}$$

- This can be written in more useful (compact) form as:

$$y_t = \sum_{i=0}^{3-1} a^i \mathbb{E}_t x_{t+i} + \color{blue}{a^3 \mathbb{E}_t y_{t+3}}$$

- Generalizing for the n-th iteration (see **Appendix B** for a step-by-step derivation of a similar process)

$$y_t = \sum_{i=0}^{n-1} a^i \mathbb{E}_t x_{t+i} + \color{blue}{a^n \mathbb{E}_t y_{t+n}} \tag{6}$$

Stability

$$y_t = \sum_{i=0}^{n-1} a^i \mathbb{E}_t x_{t+i} + a^n \mathbb{E}_t y_{t+n}$$

- To avoid explosive behavior (secure a stable equilibrium), impose the condition:

$$|a| < 1$$

- Which implies that:

$$\lim_{n \to \infty} a^n \mathbb{E}_t y_{t+n} = 0$$

- And finally we get the solution to the stable equilibrium:

$$y_t = \sum_{i=0}^{n-1} a^i \mathbb{E}_t x_{t+i}. \tag{7}$$

What determines $\mathbb{E}_t x_{t+i}$?

- The solution to eq. (7) depends on two things:
    - The forces that affect x_t
    - The type of information we have about x_t
- If x_t is a stochastic process we can compute its mean under two different assumptions:
    - We know what is the true mean of this process x_t:
        - Unconditional expectations of x_t
    - We do not know what is the true mean of x_t, but we have some observations about x_t over time:
        - Conditional expectations of x_t based on such information.
- Next we show how to compute these two expected values.

Unconditional expectations

- Suppose that x_t is given by the following stochastic process:

$$x_t = \phi + \rho x_{t-1} + \varepsilon_t \quad , \qquad \varepsilon_t \sim \mathcal{N}\left(0, \sigma^2\right)$$

- Assuming unconditional expectations, the mean is given by the (deterministic) steady-state value of x_t:

$$x_t = x_{t-1} = \overline{x}$$

- Which leads to

$$\overline{x} = \phi + \rho\overline{x} + 0 \Rightarrow \overline{x} = \frac{\phi}{1-\rho}$$

- Therefore, the expected (unconditional) value of $\mathbb{E}_t x_{t+i}$ is given by:

$$\mathbb{E}_t x_{t+i} = \overline{x} = \frac{\phi}{1-\rho} \tag{8}$$

Conditional expectations

- Apply the expectations operator to $x_t = \phi + \rho x_{t-1} + \varepsilon_t$, up to third iteration:

$$
\begin{aligned}
x_t &= \phi + \rho x_{t-1} + \varepsilon_t \\
\mathbb{E}_t x_{t+1} &= \phi + \rho \mathbb{E}_t x_t + \mathbb{E}_t \varepsilon_{t+1} = \phi + \rho x_t + 0 = \phi + \rho x_t \\
\mathbb{E}_t x_{t+2} &= \phi + \rho \mathbb{E}_t x_{t+1} + \mathbb{E}_t \varepsilon_{t+2} = \phi + \rho \left[\phi + \rho x_t\right] + 0 = \phi + \rho\phi + \rho^2 x_t \\
\mathbb{E}_t x_{t+3} &= \phi + \rho \mathbb{E}_t x_{t+2} + \mathbb{E}_t \varepsilon_{t+3} = \phi + \rho \left[\phi + \rho\phi + \rho^2 x_t\right] + 0 = \underbrace{\phi + \rho\phi + \rho^2\phi}_{=\sum_{k=0}^{3-1} \phi\rho^k} + \rho^3 x_t
\end{aligned}
$$

- Then, generalize to the ith iteration

$$
\mathbb{E}_t x_{t+i} = \sum_{k=0}^{i-1} \phi\rho^k + \rho^i x_t = \frac{\phi}{1-\rho} + \rho^i x_t \tag{9}
$$

- If we do not know any x_t, then $\mathbb{E}_t x_{t+i} = \dfrac{\phi}{1-\rho}$, which is its unconditional mean.

Solution: Conditional/Unconditional Expectations

- To get the solution to eq. (7), with unconditional expectations insert eq. (8) into (7).
- If we use conditional expectations insert eq. (9) into (7).
- The solution with unconditional expectations unconditional expectations is:

$$y_t = \sum_{i=0}^{n-1} a^i \mathbb{E}_t x_{t+i} = \sum_{i=0}^{n-1} a^i \overline{x} = \frac{\overline{x}}{1-a}$$

- The solution with conditional expectations is given by:

$$y_t = \sum_{i=0}^{n-1} a^i \mathbb{E}_t x_{t+i} = \sum_{i=0}^{n-1} a^i \left[\frac{\phi}{1-\rho} + \rho^i x_t\right] = \sum_{i=0}^{n-1} \left[a^i \frac{\phi}{1-p} + a^i \rho^i x_t\right] = \frac{\phi/(1-\rho)}{1-a} + \frac{x_t}{1-a\rho}$$

4. Summary: Stability Conditions

Predetermined variables

- A process is pre-determined if its behavior depends only upon past observations:

$$x_{t+1} = \phi + \rho x_t + \varepsilon_{t+1} \ , \quad \varepsilon \sim i.i.d.\,(0, \sigma^2)$$

- Its dynamics will be expressed at the n-th iteration b (see **Appendix C** for details):

$$x_t = \rho^n x_0 + \sum_{i=0}^{n-1} \rho^i \phi + \sum_{i=0}^{n-1} \rho^i \varepsilon_{t-i} \tag{10}$$

- If $|\rho| < 1$: stable solution
  If $|\rho| > 1$: no stable solution
  If $|\rho| = 1$: no solution

Forward-Looking Variables

- The process' behavior depends also on expected future realizations:

$$y_t = x_t + a \cdot \mathbb{E}_t y_{t+1}$$

- Its dynamics will be expressed at the n-th iteration by:

$$y_t = \sum_{i=0}^{n-1} a^i \mathbb{E}_t x_{t+i} + a^n \mathbb{E}_t y_{t+n}) \tag{11}$$

- If $|a| < 1$: stable solution
  If $|a| > 1$: no stable solution
  If $|a| = 1$: no solution

Forward-Looking Variables: a Twist

- Models with RE are difficult (if not impossible) to solve by pencil and paper.
- We have to resort to the computer to "approximate" a solution for us.
- This is done by writing the model with all $t+1$ variables on the left-hand side of the state-space representation of the system, and those with t on its right-hand side.
- In this case, we should write:

$$\mathbb{E}_t y_{t+n} = -(1/a)x_t + (1/a)y_t$$

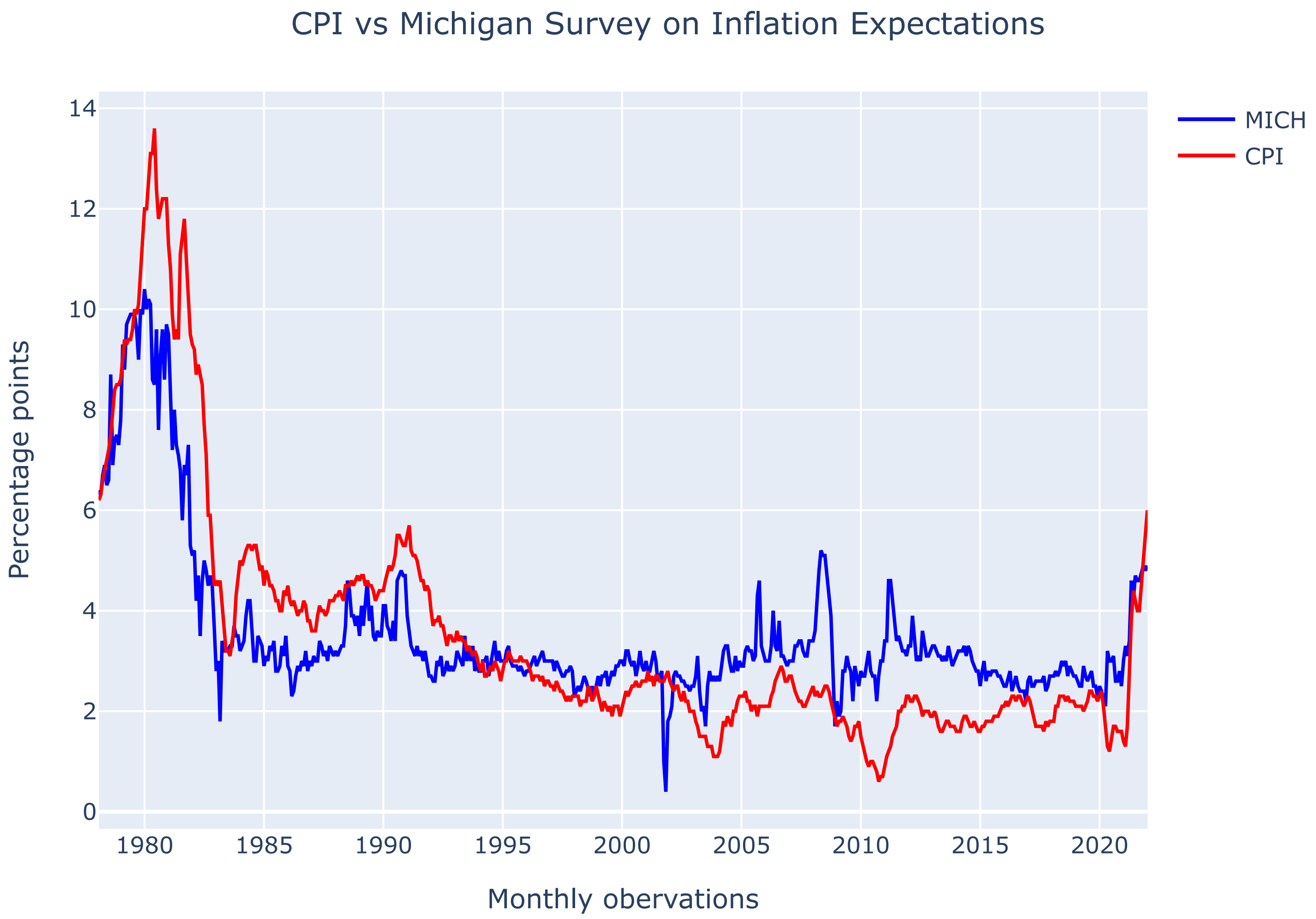
- It is easy to see that if $|a| < 1$, then $|1/a| > 1$: stability condition is the inverse.
- Therefore, if the model is written in this way, ***stability requires*** $|1/a| > 1$.

Empirical Relevance of RE

- In the early 1970s, Robert E. Lucas and Thomas Sargent published a series of papers that created a revolution in macroeconomics.
- They argued that most macroeconomic behavior depends on expectations, and adaptive expectations (AE) was a poor framework.
- The criticisms were already mentioned above:
  - AE uses only one type of information (past), which is not rational.
  - AE leads to systematic mistakes … which does not make sense.
- They proposed the concept of RE that we have been discussing here.
- How well does such a concept perform when confronted with evidence?
- Let us see if people, by using all relevant information, do not make systematic mistakes.

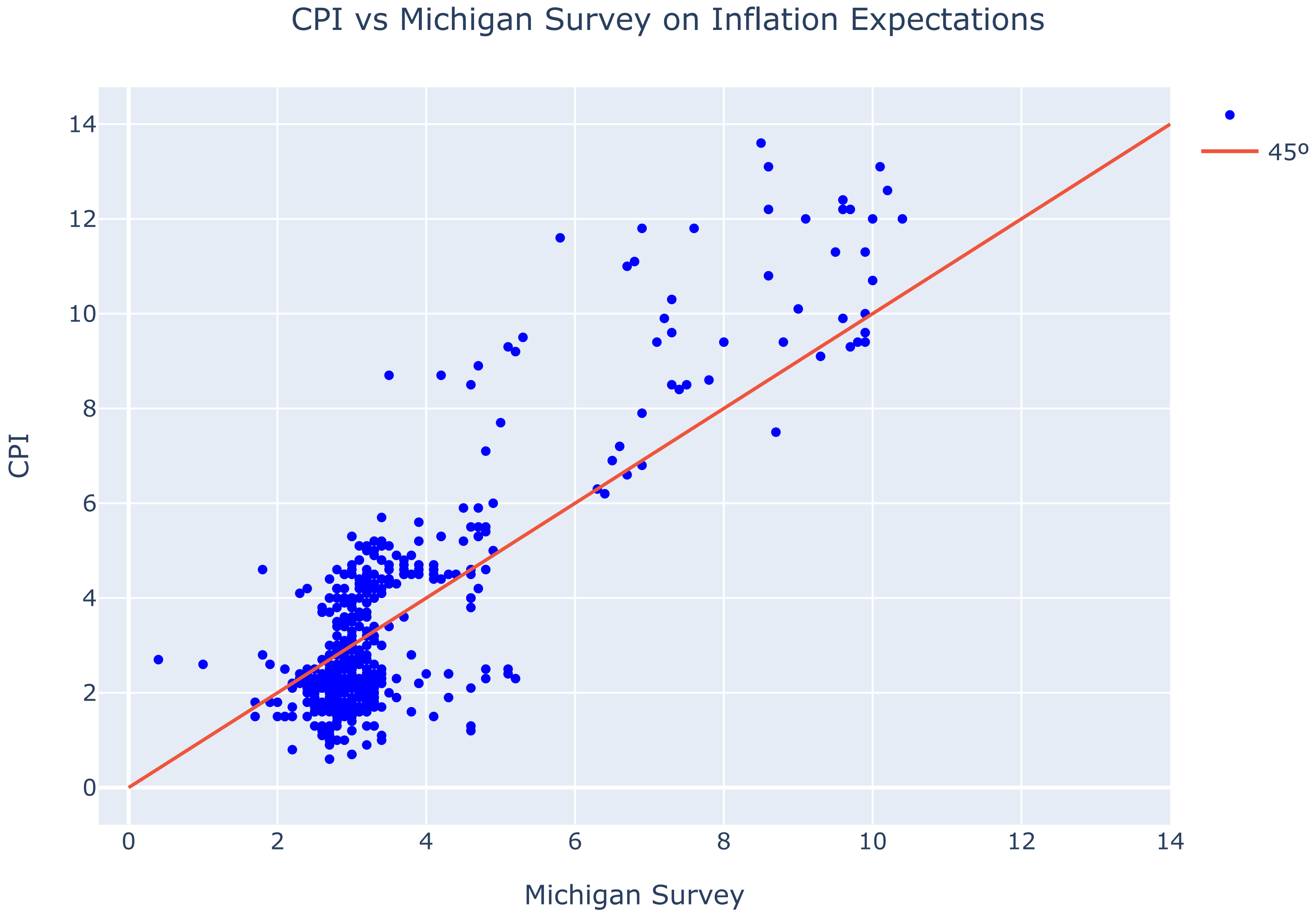
Michigan Survey on Inflation Expectations

- The most cited survey in macroeconomics.

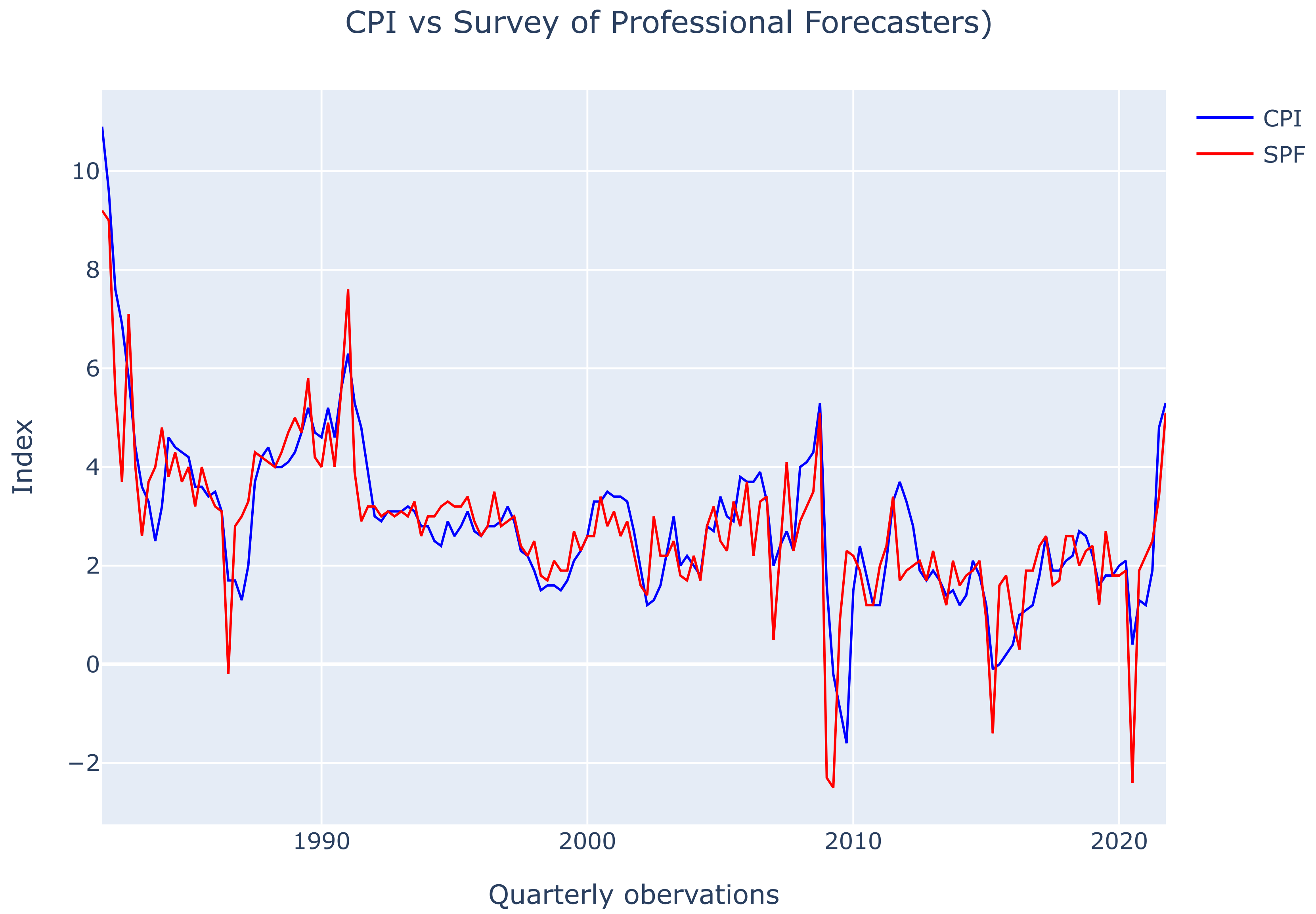


Michigan Survey on Inflation Expectations (cont.)

- Systematic mistakes in inflation expectations; MICH performs quite poorly.

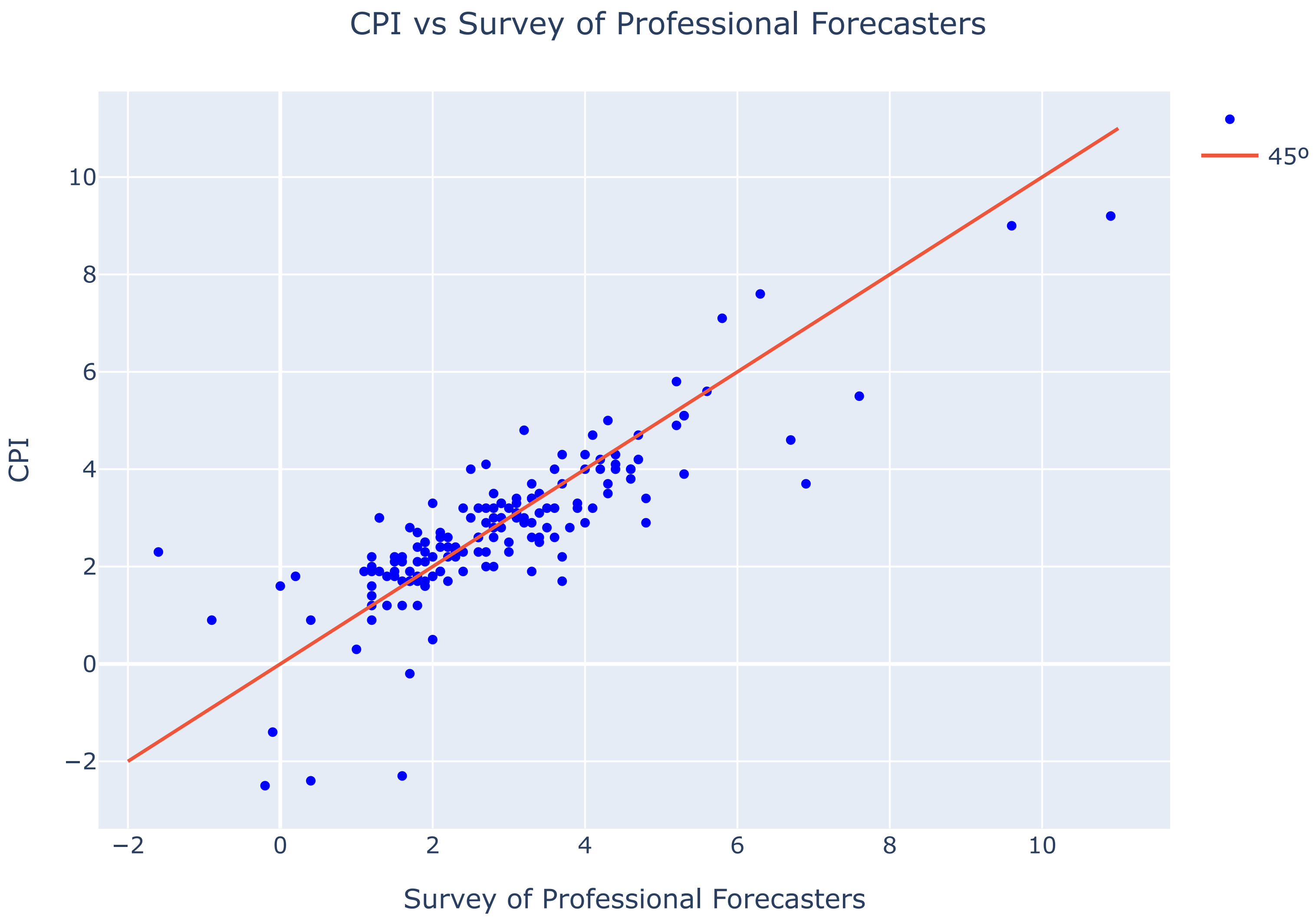


Survey of Professional Forecasters



Survey of Professional Forecasters

The SPF seems to produce unbiased expectations, and gives support to RE.



A Model with Forward & Backward-Looking Behavior

$$
\begin{aligned}
&IS: && \hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} - \tfrac{1}{\sigma}(i_t - \mathbb{E}_t \pi_{t+1} - r_t^n) + u_t \\
&AS: && \pi_t = \mu \pi_{t-1} + \kappa \hat{y}_t + \beta \mathbb{E}_t \pi_{t+1} + s_t \\
&MP: && i_t = \pi_t + r_t^n + \phi_\pi (\pi_t - \pi_t^*) + \phi_y \hat{y}_t \\
&\text{Shocks}: && r_t^n = \rho_r \cdot r_{t-1}^n + \varepsilon_t^r \ , \quad u_t = \rho_u \cdot u_{t-1} + \varepsilon_t^u \ , \quad s_t = \rho_s \cdot s_{t-1} + \varepsilon_t^s
\end{aligned}
$$

- $\{i, r_t^n, \hat{y}, \pi, u_t, s_t, \varepsilon_t\}$: nominal interest rate, natural real interest rate, output-gap, inflation rate, demand shock, supply shock, and a random disturbance.
- $\{\sigma, \mu, \kappa, \beta, \phi_\pi, \phi_y, \pi_t^*, \rho\}$ are parameters
- Forward-looking variables: $\hat{y}_t, \pi_t$
- Backward-looking variables: r_t^n, u_t, s_t
- Static variables: i_t
- How can we solve this ***New Keynesian Model***? Using a computer. Next!

5. Readings

- There is no compulsory reading for this session. We hope that the slides and the notebook will be sufficient to provide a good grasp of the two types of expectations in macroeconomics.
- Many textbooks deal with this subject in a way that is not very useful for our course. They treat this subject in an elementary way or offer a very sophisticated presentation, usually extremely mathematical but short on content.
- There is a textbook that feels quite good for our level:
  Patrick Minford and David Peel (2019). *Advanced Macroeconomics: A Primer*, Second Edition, Edward Elgar, Cheltenham.
- Chapter 2 deals extensively with AE and RE. However, the former chapter is quite long (40 pages), making it more suitable to be used as complementary material rather than as compulsory reading. But it is by far the best treatment of this subject at this level.

- Another excellent treatment of AE and RE can be found in the textbook:

  Ben J. Heijdra (2017). *Foundations of Modern Macroeconomics*. Third Edition, Oxford UP, Oxford.

- Chapter 5 deal with this topic at great length (40 pages), but the subject is discussed at a more advanced level.

Appendix A

A step-by-step derivation of **equation (2)**

$$
\begin{aligned}
& P_t^e = \alpha P_{t-1} + (1-\alpha) P_{t-1}^e \\
& \downarrow \qquad\qquad\qquad\qquad\qquad\qquad \nwarrow P_{t-1}^e = \alpha P_{t-2} + (1-\alpha) P_{t-2}^e \\
& P_t^e = \alpha P_{t-1} + (1-\alpha)\left[\alpha P_{t-2} + (1-\alpha) P_{t-2}^e\right] \\
& P_t^e = \alpha P_{t-1} + \alpha(1-\alpha) P_{t-2} + (1-\alpha)^2 P_{t-2}^e \\
& \downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow P_{t-2}^e = \alpha P_{t-3} + (1-\alpha) P_{t-3}^e \\
& P_t^e = \alpha P_{t-1} + \alpha(1-\alpha) P_{t-2} + (1-\alpha)^2\left[\alpha P_{t-3} + (1-\alpha) P_{t-3}^e\right] \\
& P_t^e = \alpha P_{t-1} + \alpha(1-\alpha) P_{t-2} + \alpha(1-\alpha)^2 P_{t-3} + (1-\alpha)^3 P_{t-3}^e \\
& P_t^e = \sum_{i=0}^{3-1} \alpha(1-\alpha)^i P_{t-1-i} + {\color{blue}(1-\alpha)^3 P_{t-3}^e}
\end{aligned}
$$

Appendix B

A step-by-step derivation of **equation (6)**

$$y_t = \alpha + \beta \mathbb{E}_t y_{t+1} + x_t$$ 1st iteration: $t \to t+1$

$$\downarrow \qquad \nwarrow \mathbb{E}_t y_{t+1} = \alpha + \beta \mathbb{E}_t y_{t+2} + \mathbb{E}_t x_{t+1}$$ Law Iterat. expectations

$$y_t = \alpha + \beta \left[\alpha + \beta \mathbb{E}_t y_{t+2} + \mathbb{E}_t x_{t+1}\right] + x_t$$

$$y_t = \alpha + \beta\alpha + \beta^2 \mathbb{E}_t y_{t+2} + \beta \mathbb{E}_t x_{t+1} + x_t$$ 2nd iteration: $t+1 \to t+2$

$$\downarrow \qquad \nwarrow \mathbb{E}_t y_{t+2} = \alpha + \beta \mathbb{E}_t y_{t+3} + \mathbb{E}_t x_{t+2}$$

$$y_t = \beta^0\alpha + \beta^1\alpha + \beta^2\alpha + \beta^3 \mathbb{E}_t y_{t+3} + \beta^2 \mathbb{E}_t x_{t+2} + \beta^1 \mathbb{E}_t x_{t+1} + \beta^0 \mathbb{E}_t x_t$$

$$y_t = \sum_{i=0}^{3-1} \beta^i \alpha + \beta^3 \mathbb{E}_t y_{t+3} + \sum_{i=0}^{3-1} \beta^i \mathbb{E}_t x_{t+i}$$ **3rd iteration: $t+2 \to t+3$**

$$y_t = \sum_{i=0}^{n-1} \beta^i \alpha + \beta^n \mathbb{E}_t y_{t+n} + \sum_{i=0}^{n-1} \beta^i \mathbb{E}_t x_{t+i}$$ **nth iteration: $\ldots \to t+n$**

Appendix C

A step-by-step derivation of **equation (10)**

$$x_t = \phi + \rho x_{t-1} + \varepsilon_t$$

1st iteration: $t \to t-1$

$$\downarrow \qquad \nwarrow \; x_{t-1} = \phi + \rho x_{t-2} + \varepsilon_{t-1}$$

Repeated substitution

$$x_t = \phi + \rho\left[\phi + \rho x_{t-2} + \varepsilon_{t-1}\right] + \varepsilon_t$$

$$x_t = \phi + \rho\phi + \rho^2 x_{t-2} + \rho\varepsilon_{t-1} + \varepsilon_t$$

2nd iteration: $t-1 \to t-2$

$$\downarrow \qquad \nwarrow \; x_{t-2} = \phi + \rho x_{t-3} + \varepsilon_{t-2}$$

$$x_t = \rho^0\phi + \rho^1\phi + \rho^2\phi + \rho^3 x_{t-3} + \rho^2\varepsilon_{t-2} + \rho^1\varepsilon_{t-1} + \rho^0\varepsilon_t$$

$$x_t = \sum_{i=0}^{3-1}\rho^i\phi + \rho^3 x_{t-3} + \sum_{i=0}^{3-1}\rho^i\varepsilon_{t-i}$$

3rd iteration: $t-2 \to t-3$

$$x_t = \sum_{i=0}^{n-1}\rho^i\phi + \rho^n x_{t-n} + \sum_{i=0}^{n-1}\rho^i\varepsilon_{t-i}$$

nth iteration: $\ldots\ t-n$